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MECHANICS BY QUATERNIONS.

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[Continued from page 24.]

THUS the distance of the center of gravity along i is equal to $\frac{3}{8}$ of the projection of εt_2 on i, along j to $3 \div 4\pi$ times the projection of εt_2 on the plane of j and k, while the distance along k is independent of t_2.

For the c. g. of the *surface* we have as before

$$\frac{1-u^4}{1-u^3} = \frac{1-u+u^2+u^3}{1+u+u^2} = \frac{4}{3},$$

where $u = 1$; which value is to be substituted in the above equations.

The volume whose c. g. is given by the general equation above is bounded by two similar hyperboloids, two circular cones having their vertices at the origin and for directrices the two circles

$${}_{,}\rho = i^{\theta\div\pi}(bj+t_1\varepsilon)i^{-\theta\div\pi} \text{ and } {}_{,}\rho = i^{\theta\div\pi}(bj+t_2\varepsilon)i^{-\theta\div\pi},$$

and two planes passing respectively through the origin and the generatrix in its initial position, and in its position after turning thro' the angle θ_1.

(31) *Friction.*—If two bodies, being in contact, have motion relatively to each other a resistance to this relative motion is developed which is called *friction.* If one body rolls on the other the resistance is called *rolling* friction, if it slides, the resistance is called *sliding* friction. The latter only will be here considered.

Sliding friction is generally assumed to be,

(*a*). Directly proportional to the normal pressure between the bodies:

(*b*). Independent of the area of the surface of contact:

(*c*). Independent of the relative velocity of the two bodies.

These laws were originally stated as the results of experiment, but later experiment has shown that the last, at least, is not strictly true. Within reasonable limits however they are probably near enough to the truth for practical purposes. The friction we have to consider here is that called into play when a body is just at the point of beginning to move upon another; this may be slightly greater than the friction after motion has been established.

First consider a particle resting against a rough surface in such a way that the forces acting on it press it against the surface.

Let ${}_{,}\nu$ be a vector along the normal to the surface, ${}_{,}R$ the resultant of all the forces acting on the particle except friction and the normal pressure of the surface against it, ${}_{,}N$ this normal pressure, and ${}_{,}F$ the friction called into play. Then, as in equation (5), we shall have for equilibrium

$$V{}_{,}\nu({}_{,}R+{}_{,}F+{}_{,}N) = V{}_{,}\nu({}_{,}R+{}_{,}F) = 0, \tag{73}$$

the term $V{}_{,}\nu{}_{,}N$ being dropped because equal to zero. Also since ${}_{,}F$ acts tangentially to the surface, we must hve $S{}_{,}\nu{}_{,}F = 0$, therefore by (73)

$${}_{,}\nu{}_{,}F = S{}_{,}\nu{}_{,}F + V{}_{,}\nu{}_{,}F = -V{}_{,}\nu{}_{,}R, \text{ or } {}_{,}F = -{}_{,}\nu^{-1}\,V{}_{,}\nu{}_{,}R. \tag{74}$$

Equation (74) expresses the friction actually called into play so long as the particle does not move. By the first law given above the greatest am't of friction that can be called into play is a multiple of the normal pressure, i. e., $F = \mu N$, in which μ is a numerical multiplier determined experimentally and called the coefficient of friction.

The normal pressure must be equal and opposite to the component of ${}_{,}R$ along ${}_{,}\nu$, i. e., ${}_{,}N = -{}_{,}\nu^{-1}S{}_{,}\nu{}_{,}R$; therefore

$$F = \mu N = \mu T{}_{,}N = \mp\, \mu S{}_{,}\nu{}_{,}R.T{}_{,}\nu^{-1},$$

in which the upper or lower sign is taken according as the angle between ${}_{,}\nu$ and ${}_{,}R$ is acute or obtuse. Comparing this value of F with (74) we have

$$TV{}_{,}\nu{}_{,}R \pm \mu S{}_{,}\nu{}_{,}R = 0, \tag{75}$$

as the limiting condition of equilibrium when the possible friction is all called into play, and the particle is at the point of starting.

The particle will remain at rest so long as $TV{}_{,}\nu{}_{,}R$ is not numerically greater than $\mu S{}_{,}\nu{}_{,}R$. If ${}_{,}R$ be regarded as variable and ${}_{,}\nu$ constant eq'n (75) becomes the equation of a cone of revolution about ${}_{,}\nu$ as axis, within, or on the surface of which ${}_{,}R$ must remain if the particle is not to move.

If the surface be a plane inclined at the angle α at which the particle is at the point of starting; if $\varepsilon = U{}_{,}\nu =$ vector perpendicular to the plane, drawn so as to make an acute angle with ${}_{,}W$, the weight of the particle, and ${}_{,}R = {}_{,}W$; then by (75) $TV\varepsilon{}_{,}W + \mu S\varepsilon{}_{,}W = 0$, or

$$\mu = -\frac{TV\varepsilon{}_{,}W}{S\varepsilon{}_{,}W} = \tan\alpha. \tag{76}$$

α is called the angle of friction.

Particle Constrained to Remain on a Rough Curve.—We have by Equation (4), since $S{}_{,}N{}_{,}\varphi'(t) = 0$,

$$S({}_{,}R+{}_{,}F){}_{,}\psi'(t) = 0. \tag{77}$$

Hence, as $V{}_{,}F{}_{,}\varphi'(t) = 0$, we have ${}_{,}F{}_{,}\varphi'(t) = -S{}_{,}R{}_{,}\varphi'(t)$; therefore

$${}_{,}F = -[{}_{,}\varphi'(t)]^{-1}S{}_{,}R{}_{,}\varphi'(t)\text{: also } {}_{,}N = -[{}_{,}\varphi'(t)]^{-1}\,V{}_{,}\varphi'(t){}_{,}R.$$

$$\therefore\ F = \mu N = \mu TV{}_{,}\varphi'(t){}_{,}R.T[{}_{,}\varphi'(t)]^{-1} = \mp\, S{}_{,}R{}_{,}\varphi'(t).T[{}_{,}\varphi'(t)]^{-1},$$

in the limiting case, when F has its greatest value. The signs are to be taken, as before, so as to make the right hand member positive. We have thus for equilibrium.

$$\mu TV{}_{,}R{}_{,}\varphi'(t) \pm S{}_{,}R{}_{,}\varphi'(t) > \text{ or } = 0. \tag{78}$$

This shows that, if the particle is to remain at rest, ${}_\prime R$ must be outside the circular cone whose equation is

$$\mu TV_\prime\rho_\prime\varphi'(t) \pm S_\prime\rho_\prime\varphi'(t) = 0,$$

in which t is taken constant, and ${}_\prime\varphi'(t)$ is a vector along the axis, or else on the surface of this cone.

(32). *Equilibrium of Inextensible Flexible Strings.*—By a flexible string will be understood one which offers no resistance whatever to bending.

The cross-section of the strings treated will be regarded as very small in comparison with their length. Let ${}_\prime P$ be the pull or tension at any point of the string. It will evidently act tangentially to the string, so that

$$U_\prime P = Ud_\prime\rho = \frac{d_\prime\rho}{Td_\prime\rho} = \frac{d_\prime\rho}{ds} = {}_\prime\rho'.$$

In treating this subject we shall use *primes* to indicate differentiation with respect to s, the length of any portion of the string, as above; thus ${}_\prime P' = d_\prime P \div ds$, etc. ${}_\prime P'$ is then the rate of variation of ${}_\prime P$ as we go along the string, and, if this be multiplied by δs, we have the total variation of ${}_\prime P$ in the infinitesimal distance δs.

In order that the element δs may be in equilibrium the sum of all the forces acting on it must be zero. Let ${}_\prime R\delta s$ be the resultant of all the exterior forces acting on the element: then the other forces are ${}_\prime P$ at one end, and $-{}_\prime P + {}_\prime P'\delta s$ at the other; therefore

$${}_\prime P + {}_\prime R\delta s - {}_\prime P + {}_\prime P'\delta s = 0, \text{ or } {}_\prime P' + {}_\prime R = 0. \tag{79}$$

Whence
$${}_\prime P = -\int{}_\prime R ds. \tag{80}$$

As we have seen above that $U_\prime P = {}_\prime\rho'$, we have

$${}_\prime P' = \frac{d}{ds}\left(P_\prime\rho'\right) = P'_\prime\rho' + P_\prime\rho'' = -{}_\prime R. \tag{81}$$

If ${}_\prime r$ is the vector radius of curvature of the string at the end of ${}_\prime\rho$, we have (Tait's Quat., Art. 283)

$${}_\prime\rho'' = -{}_\prime r^{-1};\ \therefore\ P'_\prime\rho' - P_\prime r^{-1} = -{}_\prime R. \tag{82}$$

If P be constant, so that $P' = 0$, this equation shows that ${}_\prime R$ acts along the principal normal to the curve, and that R varies as $1 \div r$. Operate on (81) by $S.V_\prime R_\prime\rho'$;

$$\therefore\ S_\prime R_\prime\rho'{}_\prime\rho'' = 0, \tag{83}$$

which shows that ${}_\prime R$ always lies on the osculating plane of the curve.

Operate on (81) by $S.{}_\prime\rho'$; then, since $T_\prime\rho' = 1$ and $S_\prime\rho'{}_\prime\rho'' = 0$ (Tait's Quat., Art. 282), we have

$$P' = S_\prime R_\prime\rho', \text{ or } P = \int S_\prime R d_\prime\rho. \tag{84}$$

Next operate on (81) by $S.{}_\prime\rho''$; $\therefore\ P_\prime\rho''^2 = -S_\prime R_\prime\rho''$, or

$$P = -S_\prime R_\prime\rho''^{-1} = +S_\prime R_\prime r. \tag{85}$$

Comparing (84) with (85) we have

$$S{}_{,}R{}_{,}\rho''^{-1} = -\int S{}_{,}Rd{}_{,}\rho. \tag{86}$$

This being a relation between ${}_{,}\rho$ and ${}_{,}R$ independent of ${}_{,}P$ is the equation of the curve of the string.

Suppose as a particular case that the string lies in the plane of i and j, and that ${}_{,}R$ is parallel to j; so that $Sk{}_{,}\rho = Sk{}_{,}\rho' = Sk{}_{,}R = 0$, and $Si{}_{,}R = 0$, or ${}_{,}R = \pm Rj$. Then by (79)

$$Si{}_{,}P' = 0\,;\ \therefore\ Si{}_{,}P = PSi{}_{,}\rho' = \text{constant} = -H, \text{ say,}$$

$$\therefore\ P = -H \div Si{}_{,}\rho'. \tag{87}$$

It appears therefore that when ${}_{,}R$ is parallel to j the horizontal component of the tension is always constant. Again by (79)

$$Sj{}_{,}P' = Sj\frac{d}{ds}(P{}_{,}\rho') = \frac{d}{ds}(PSj{}_{,}\rho') = -H\frac{d}{ds}\left(\frac{Sj{}_{,}\rho'}{Si{}_{,}\rho'}\right) = -Sj{}_{,}R = \pm R. \tag{88}$$

The upper or lower sign of the last member is to be taken according as ${}_{,}R$ acts upward or downward.

As another special case, let the force act from or towards a fixed point. If the origin be taken at this point we shall have ${}_{,}R = \pm RU{}_{,}\rho$, according as the force acts from or towards the origin.

By (83) $S{}_{,}\rho{}_{,}\rho'{}_{,}\rho'' = 0$, which shows that the string lies in a plane thro' the origin. By (84) and (85)

$$P = \pm\int RSU{}_{,}\rho.d{}_{,}\rho = \mp\int RSd{}_{,}\rho.U{}_{,}\rho^{-1} = \mp\int RdT{}_{,}\rho = \mp\int Rd\rho$$

$$= \mp RSU{}_{,}\rho.{}_{,}\rho''^{-1}. \tag{89}$$

This is the equation of the curve of the string. We may however derive the equation in a form not involving ${}_{,}\rho''$. We have by (79)

$${}_{,}P' = -{}_{,}R = \mp RU{}_{,}\rho\,;\ \therefore\ V{}_{,}\rho{}_{,}P' = 0.$$

Now $V{}_{,}\rho'{}_{,}P = PV{}_{,}\rho'{}_{,}\rho' = 0$; $\therefore$ adding and integrating $V{}_{,}\rho{}_{,}P = {}_{,}c$;

$$\therefore\ TV{}_{,}\rho{}_{,}P = PTV{}_{,}\rho{}_{,}\rho' = c,$$

whence by comparison with (89)

$$P = c(TV{}_{,}\rho{}_{,}\rho')^{-1} = \mp\int Rd\rho. \tag{90}$$

This equation gives by integration the curve of the string, and will give the same curve whether the upper or lower sign of the right hand member be taken. The sign ought to be so taken however as to make the integral $\int Rd\rho$ positive.

(33). Let us now apply our general equation to the case of a string stretched over a curve. We will first suppose the curve smooth and the string without weight. On account of the smoothness of the curve the only action possible between it and the string will be in the direction of the principal normal to the curve at the point considered. Call this action $-NU_{,}r$, minus because ${}_{,}r$ is directed inwards. Then

$$ {}_{,}R = -NU_{,}r = -NU_{,}\rho'', $$

and by (81)

$$ P'{}_{,}\rho' + P_{,}\rho'' = NU_{,}\rho'', \text{ or } P'{}_{,}\rho' = (N - Pr^{-1})U_{,}\rho''. $$

This equation can only be satisfied by equating to zero separately the coefficients of ${}_{,}\rho'$ and $U_{,}\rho''$. Thus we have

$$ P = \text{constant, and } N = Pr^{-1}. \tag{91} $$

The total pressure on the curve will be

$$ \int N ds = \int Pr^{-1} ds. \tag{92} $$

If we consider the weight of the string, we shall have ${}_{,}R = -NU_{,}\rho'' - Wj$, if j be a vertical unit vector and W be the weight of a unit of length of the string. By (84) $P' = S_{,}R_{,}\rho' = -WSj_{,}\rho'$; and by (85) $P = -S_{,}R_{,}\rho''^{-1} = S(NU_{,}\rho'' + Wj)_{,}\rho''^{-1} = rN - WSj_{,}r$: whence

$$ P = rN - WSj_{,}r = -W\int Sjd_{,}\rho. \tag{93} $$

As the equation of the curve is known ${}_{,}r$ and $d_{,}\rho$ are known, and therefore P and N may be found by (93).

Finally, suppose the curve to be rough, then on each element $\delta_{,}\rho$ of the string there will act a force of friction equal to $-\mu NUd_{,}\rho = -\mu N_{,}\rho'$; so that ${}_{,}R = -NU_{,}\rho'' - Wj - \mu N_{,}\rho'$. Hence by (84)

$$ P' = S_{,}R_{,}\rho' = -WSj_{,}\rho' + \mu N. \tag{94} $$

By (85)

$$ P = S_{,}R_{,}r = -WSj_{,}r + rN. \tag{95} $$

Eliminating N between (94) and (95) we have

$$ P' - \frac{\mu}{r}P = WSj(\mu U_{,}\rho'' - {}_{,}\rho'), \tag{96} $$

a differential equation for determining P. These equations hold for tortuous as well as plane curves.

We will next consider the case of a string stretched over a smooth surface. The action between the string and surface must be normal to the surface; therefore let $-NU_{,}\nu$ be this normal pressure, and let ${}_{,}Q$ be the resultant of all the other forces acting on the string per unit of length.

Then ${}_{,}R = {}_{,}Q - NU_{,}\nu$, which in (86) gives

$$ S({}_{,}Q - NU_{,}\nu)_{,}\rho''^{-1} = -\int S_{,}Qd_{,}\rho. $$

But by (83)

$$ S_{,}\rho'{}_{,}\rho''{}_{,}R = 0 = S_{,}\rho'{}_{,}\rho''({}_{,}Q - NU_{,}\nu); $$

$$ \therefore N = \frac{S_{,}\rho'{}_{,}\rho''{}_{,}Q}{S_{,}\rho'{}_{,}\rho'' U_{,}\nu}. $$

Substituing this value of N in the previous equation and reducing we have

$$S\left({}_{,}Q+{}_{,}\rho''\int S_{,}Qd_{,}\rho\right){}_{,}\rho'{}_{,}\upsilon = 0, \tag{97}$$

a differential equation for determining the curve. If ${}_{,}Q$ is normal to the surface and $= C_{,}\upsilon$, say, then $S_{,}Q_{,}\rho'{}_{,}\upsilon = 0$, and therefore $S_{,}\rho'{}_{,}\rho''{}_{,}\upsilon = 0$, so that the normal to the surface lies in the osculatory plane of the curve, a property of the shortest line that can be drawn on the surface between two points of it.

(34). *Extensible Strings.*—Under the action of a force, small compared with that which will break it, a string is found to be stretched to an amount very approximately proprtional to the applied force. Thus if s is the original length of a straight string and the forces P_1 and P_2, applied at one end in the direction of its length, stretch it to the lengths s_1 and s_2 respectively, the above law is expressed by the equation

$$\frac{s_1 - s}{s_2 - s} = \frac{P_1}{P_2}.$$

If this law be supposed to hold for forces of *any* magnitude, and P_2 be so taken as to stretch s to double its original length, so that $s_2 = 2s$, then P_2 is called the *modulus of elasticity*, and is usually represented by E. If we call the stretched length of the string σ, and drop the suffix of the P_1, our formula then becomes

$$\sigma - s = sPE^{-1}. \tag{98}$$

All the equations previously found apply equally well to extensible st'gs if we consider them after the extension has occurred. Thus (79) becomes

$$\frac{d_{,}P}{d\sigma} = -{}_{,}R_1, \tag{99}$$

in which ${}_{,}R_1$ is written instead of ${}_{,}R$ because the mass of a unit of length of the string is changed by the stretching. To express ${}_{,}R_1$ in terms of the mass before stretching, let ${}_{,}R = m_{,}q$, and ${}_{,}R_1 = m_1{}_{,}q$, and let δs and $\delta\sigma$ be the original and stretched lengths of an element of the string. Then $m\delta s = m_1\delta\sigma$, and by (98) $\delta\sigma = \delta s(1 + PE^{-1})$; $\therefore m = m_1(1+PE^{-1})$, and ${}_{,}R_1 = m_{,}q(1+PE^{-1})^{-1}$, whence

$$\frac{d_{,}P}{d\sigma} = -m_{,}qE(P+E)^{-1}. \tag{100}$$

(35). *Examples.*—1st. Common Catenary. This being the curve of a uniform string hanging freely and acted on by its own weight only, we shall have $R = -wj$, in which w is the weight per unit of length.

By (79) ${}_{,}P' = wj$; $\quad\therefore {}_{,}P = wjs + Hi$, (101) in which Hi is the constant of integration, being the value of ${}_{,}P$ when $s=0$ if s be measured from the lowest point of the string. Taking tensors

$$P = \sqrt{(w^2s^2 + H^2)}.$$

By (84) $\qquad P = -w\int Sjd_{,}\rho = -wSj_{,}\rho + H,$

in which H is the value of P already found when ${}_{,}\rho = 0$. Put $H = hw$, so that H is measured in terms of the weight per unit of length of the string, and equate the values of P just found; hence

$$s^2 = S^2j_{,}\rho - 2hSj_{,}\rho. \tag{102}$$

By (88) $R = w = hw\,\frac{d}{ds}\left(\frac{Sj_{,}\rho'}{Si_{,}\rho'}\right) = hw\,\frac{d}{ds}\left(\tan\varphi\right)$, if φ is the angle between the tangent to the curve and the vector i. Hence $\tan\varphi = s \div h$, no constant being required if $s = 0$ when $\varphi = 0$. This is the *intrinsic* eq. of the catenary. We have also

$$\frac{d(\tan\varphi)}{ds} = \frac{d(\tan\varphi)}{ds}\cdot\frac{Sid_{,}\rho}{Sid_{,}\rho} = \frac{d(\tan\varphi)}{Sid_{,}\rho}\,.Si_{,}\rho' = -\frac{\sec^2\varphi d\varphi}{Sid_{,}\rho}\cos\varphi.$$

Hence $-h\sec\varphi d\varphi = Sid_{,}\rho$, and integrating,

$$-Si_{,}\rho = h\log(\sec\varphi+\tan\varphi) = h\log\left[(s\div h)+\sqrt{1+(s\div h)^2}\,\right]. \tag{103}$$

The constant will be zero if $Si_{,}\rho = 0$, when $\varphi = 0$.

If in (102) the origin be taken at a distance h below the lowest point of the curve instead of at that point, the equation becomes

$$s^2 = -h^2 + S^2j_{,}\rho. \tag{104}$$

If s be eliminated between (103) and (104) we have

$$-Sj_{,}\rho = \tfrac{1}{2}h\left[e^{h^{-1}Si_{,}\rho} + e^{-h^{-1}Si_{,}\rho}\right]. \tag{105}$$

2nd. Let the string be such that portions whose horizontal projections are equal have equal weights. Then ${}_{,}R = CjSi_{,}\rho'$, and by (88)

$$H\frac{d}{ds}\left(\frac{Sj_{,}\rho'}{Si_{,}\rho'}\right) = -CSi_{,}\rho',$$

whence $\qquad HSj_{,}\rho' = -CSi_{,}\rho Si_{,}\rho',$

and $\qquad 2HSj_{,}\rho = -CS^2i_{,}\rho. \qquad (106)$

There will be no constant in either integration if the origin be taken at the lowest point of the string. The equation is that of a parabola, on the condition presupposed that $Sk_{,}\rho = 0$. Without this condition it would be that of a parabolic cylinder.

Also by (79) $\qquad {}_{,}P' = -{}_{,}R = -CjSi_{,}\rho';$

$$\therefore\ {}_{,}P = -CjSi_{,}\rho + Hi,$$

in which, as before, Hi is the value of ${}_{,}P$ at the lowest point.